Complex IJ numbers

https://maalchemist.ru/CIJ/

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1 Definition

$$z = (a + ib) + j(c + id) = a + ib + jc + ijd = r * e^{i\phi} + j \cdot \rho * e^{i\psi}$$

$$i^2 \quad = \quad -1$$
$$j^2 \quad = \quad -1$$
$$(ij)^2 = \quad 1$$
$$i \cdot j \quad = j \cdot i$$

$$e^{i\phi} \quad = \cos\phi + i \cdot \sin\phi$$
$$e^{j\phi} \quad = \cos\phi + j \cdot \sin\phi$$
$$e^{ij\phi} = \cosh\phi + ij \cdot \sinh\phi$$

$$a = r * \cos\phi$$
$$b = r * \sin\phi$$
$$c = \rho * \cos\psi$$
$$d = \rho * \sin\psi$$

2 Addition and subtraction

$$u = (a_1 + ib_1) + j(c_1 + id_1)$$
$$v = (a_2 + ib_2) + j(c_2 + id_2)$$

$$z = u + v = (a + ib) + j(c + id)$$
$$a = a_1 + a_2$$
$$b = b_1 + b_2$$
$$c = c_1 + c_2$$
$$d = d_1 + d_2$$

$$z = u - v = (a + ib) + j(c + id)$$
$$a = a_1 - a_2$$
$$b = b_1 - b_2$$
$$c = c_1 - c_2$$
$$d = d_1 - d_2$$

3 Multiplication

$$u = (a_1 + ib_1) + j(c_1 + id_1) = a_1 + ib_1 + jc_1 + ijd_1$$
$$v = (a_2 + ib_2) + j(c_2 + id_2) = a_2 + ib_2 + jc_2 + ijd_2$$

$$z = u * v = (a + ib) + j(c + id)$$
$$a = a_1 * a_2 - b_1 * b_2 - c_1 * c_2 + d_1 * d_2$$
$$b = a_1 * b_2 + b_1 * a_2 - c_1 * d_2 - d_1 * c_2$$
$$c = a_1 * c_2 - b_1 * d_2 + c_1 * a_2 - d_1 * b_2$$
$$d = a_1 * d_2 + b_1 * c_2 + c_1 * b_2 + d_1 * a_2$$

$$(a + \quad ib) * (a - \quad ib) = a^2 + b^2$$
$$(a + \quad jb) * (a - \quad jb) = a^2 + b^2$$
$$(a + ijb) * (a - ijb) = a^2 - b^2$$

$$(a + ib + jc + ijd) * (a + ib + jc + ijd) = a^2 - b^2 - c^2 + d^2 + i \cdot 2(ab - cd) + j \cdot 2(ac - bd) + ij \cdot 2(ad + bc) = z^2$$

$$(a + ib + jc + ijd) * (a + ib - jc - ijd) = a^2 - b^2 + c^2 - d^2 + i \cdot 2(ab + cd)$$
$$(a + ib + jc + ijd) * (a - ib + jc - ijd) = a^2 + b^2 - c^2 - d^2 + j \cdot 2(ac + bd)$$
$$(a + ib + jc + ijd) * (a - ib - jc + ijd) = a^2 + b^2 + c^2 + d^2 + ij \cdot 2(ad - bc)$$

$$(a + ib + jc + ijd) * (a + ib + jc - ijd) = a^2 - b^2 - c^2 - d^2 + i \cdot 2ab + j \cdot 2ac + ij \cdot 2bc$$
$$(a + ib + jc + ijd) * (a + ib - jc + ijd) = a^2 - b^2 + c^2 + d^2 + i \cdot 2ab - j \cdot 2bd + ij \cdot 2ad$$
$$(a + ib + jc + ijd) * (a - ib + jc + ijd) = a^2 + b^2 - c^2 + d^2 - i \cdot 2cd + j \cdot 2ac + ij \cdot 2ad$$
$$(a + ib + jc + ijd) * (a - ib - jc - ijd) = a^2 + b^2 + c^2 - d^2 + i \cdot 2cd + j \cdot 2bd - ij \cdot 2bc$$

4 Reciprocal

4.1 <ij>

$$(a + ib + jc + ijd) * (a - ib - jc + ijd) = a^2 + b^2 + c^2 + d^2 + ij \cdot 2(ad - bc)$$

$$\frac{1}{a + ib + jc + ijd} = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d} = \frac{a - ib - jc + ijd}{a^2 + b^2 + c^2 + d^2 + ij \cdot 2(ad - bc)}$$

$$
\begin{aligned}
V &= a^2 + b^2 + c^2 + d^2 & V \quad - U \ &= (a - d)^2 + (b + c)^2 \\
U &= 2(ad - bc) & V \quad + U \ &= (a + d)^2 + (b - c)^2 \\
F &= 1 / (V^2 - U^2) & V^2 - U^2 \ &= \big((a - d)^2 + (b + c)^2\big) * \big((a + d)^2 + (b - c)^2\big)
\end{aligned}
$$

$$\frac{1}{a + ib + jc + ijd} = \frac{a - ib - jc + ijd}{V + ijU} = \frac{(a - ib - jc + ijd) * (V - ijU)}{V^2 - U^2}$$

$$(a - ib - jc + ijd) * (V - ijU) = (aV - dU) + i \cdot (-bV - cU) + j \cdot (-bU - cV) + ij \cdot (-aU + dV)$$

$$
\begin{aligned}
\underline{a} &= F * (\quad\ aV - dU) = F * (\quad\ aV - dU) \\
\underline{b} &= F * (- bV - cU) = F * (- bV - cU) \\
\underline{c} &= F * (- bU - cV) = F * (- cV - bU) \\
\underline{d} &= F * (- aU + dV) = F * (\quad\ dV - aU)
\end{aligned}
$$

$$
\begin{aligned}
\underline{a} + \underline{d} &= \quad\ F * (V * (a + d) - U * (a + d)) = \quad\ F * (V - U) * (a + d) \\
\underline{b} - \underline{c} &= - F * (V * (b - c) - U * (b - c)) = - F * (V - U) * (b - c) \\
\underline{a} - \underline{d} &= \quad\ F * (V * (a - d) + U * (a - d)) = \quad\ F * (V + U) * (a - d) \\
\underline{b} + \underline{c} &= - F * (V * (b + c) + U * (b + c)) = - F * (V + U) * (b + c)
\end{aligned}
$$

4.2 <i>

$$(a + ib + jc + ijd) * (a + ib - jc - ijd) = a^2 - b^2 + c^2 - d^2 + i \cdot 2(ab + cd)$$

$$\frac{1}{a + ib + jc + ijd} = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d} = \frac{a + ib - jc - ijd}{a^2 - b^2 + c^2 - d^2 + i \cdot 2(ab + cd)}$$

$$
\begin{aligned}
V &= a^2 - b^2 + c^2 - d^2 \\
U &= 2(ab + cd) \\
F &= 1 / (V^2 + U^2)
\end{aligned}
$$

$$\frac{1}{a + ib + jc + ijd} = \frac{a + ib - jc - ijd}{V + iU} = \frac{(a + ib - jc - ijd) * (V - iU)}{V^2 + U^2}$$

$$(a + ib - jc - ijd) * (V - iU) = (aV + bU) + i \cdot (-aU + bV) + j \cdot (-cV - dU) + ij \cdot (cU - dV)$$

$$
\begin{aligned}
\underline{a} &= F * (\quad\ aV + bU) = F * (\quad\ aV + bU) \\
\underline{b} &= F * (- aU + bV) = F * (\quad\ bV - aU) \\
\underline{c} &= F * (- cV - dU) = F * (- cV - dU) \\
\underline{d} &= F * (\quad\ cU - dV) = F * (- dV + cU)
\end{aligned}
$$

4.3 <j>

$$(a + ib + jc + ijd) * (a - ib + jc - ijd) = a^2 + b^2 - c^2 - d^2 + j \cdot 2(ac + bd)$$

$$\frac{1}{a + ib + jc + ijd} = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d} = \frac{a - ib + jc - ijd}{a^2 + b^2 - c^2 - d^2 + j \cdot 2(ac + bd)}$$

$$
\begin{aligned}
V &= a^2 + b^2 - c^2 - d^2 \\
U &= 2(ac + bd) \\
F &= 1 / (V^2 + U^2)
\end{aligned}
$$

$$\frac{1}{a + ib + jc + ijd} = \frac{a - ib + jc - ijd}{V + jU} = \frac{(a - ib + jc - ijd) * (V - jU)}{V^2 + U^2}$$

$$(a - ib + jc - ijd) * (V - jU) = (aV + cU) + i \cdot (-bV - dU) + j \cdot (-aU + cV) + ij \cdot (bU - dV)$$

$$
\begin{aligned}
\underline{a} &= F * (\quad\ aV + cU) = F * (\quad\ aV + cU) \\
\underline{b} &= F * (- bV - dU) = F * (- bV - dU) \\
\underline{c} &= F * (- aU + cV) = F * (\quad\ cV - aU) \\
\underline{d} &= F * (\quad\ bU - dV) = F * (- dV + bU)
\end{aligned}
$$

4.4 Exceptions

$z \quad = a + ib + jc + ijd = (a + ib) + j(c + id)$
$1/z = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d}$

$\underline{a} = F * (\quad aV - dU)$
$\underline{b} = F * (- bV - cU)$
$\underline{c} = F * (- cV - bU)$
$\underline{d} = F * (\quad dV - aU)$

$V = a^2 + b^2 + c^2 + d^2$
$U = 2 \cdot (ad - bc)$
$F = 1 / (V^2 - U^2)$

$V \quad - U \quad = (a - d)^2 + (b + c)^2$
$V \quad + U \quad = (a + d)^2 + (b - c)^2$
$V^2 - U^2 = \left((a - d)^2 + (b + c)^2\right) * \left((a + d)^2 + (b - c)^2\right)$

1) $a = d \neq 0 \;\wedge\; b = -c \neq 0$
    $z = a(1 + ij) + b(i - j) = a + ib - jb + ija = (a + ib) + j(-b + ia)$

2) $a = -d \neq 0 \;\wedge\; b = c \neq 0$
    $z = a(1 - ij) + b(i + j) = a + ib + jb - ija = (a + ib) + j(b - ia)$

3) $a = d = 0 \;\wedge\; b = c \neq 0$
    $z = b(i + j) = ib + jb$

4) $a = d = 0 \;\wedge\; b = -c \neq 0$
    $z = b(i - j) = ib - jb$

5) $b = c = 0 \;\wedge\; a = d \neq 0$
    $z = a(1 + ij) = a + ija$

6) $b = c = 0 \;\wedge\; a = -d \neq 0$
    $z = a(1 - ij) = a - ija$

7) $a = b = c = d = 0$
    $z = 0$

5 Conjugate $< ij >$

5.1

$$
\begin{aligned}
z &= (a+ib) + j(c+id) = a + ib + jc + ijd \\
\bar{z}_i &= (a-ib) + j(c-id) = a - ib + jc - ijd \\
\bar{z}_j &= (a+ib) - j(c+id) = a + ib - jc - ijd \\
\bar{z}_{ij} &= (a-ib) - j(c-id) = a - ib - jc + ijd
\end{aligned}
$$

5.2

$$
\begin{aligned}
z * \bar{z}_i &= (a+ib+jc+ijd) * (a-ib+jc-ijd) = a^2 + b^2 - c^2 - d^2 + j \cdot 2(ac+bd) \\
z * \bar{z}_j &= (a+ib+jc+ijd) * (a+ib-jc-ijd) = a^2 - b^2 + c^2 - d^2 + i \cdot 2(ab+cd) \\
z * \bar{z}_{ij} &= (a+ib+jc+ijd) * (a-ib-jc+ijd) = a^2 + b^2 + c^2 + d^2 + ij \cdot 2(ad-bc)
\end{aligned}
$$

$$
\begin{aligned}
\bar{z}_{ij} * \bar{z}_i &= (a-ib-jc+ijd) * (a-ib+jc-ijd) = a^2 - b^2 + c^2 - d^2 - i \cdot 2(ab+cd) \\
\bar{z}_{ij} * \bar{z}_j &= (a-ib-jc+ijd) * (a+ib-jc-ijd) = a^2 + b^2 - c^2 - d^2 - j \cdot 2(ac+bd) \\
\bar{z}_i * \bar{z}_j &= (a-ib+jc-ijd) * (a+ib-jc-ijd) = a^2 + b^2 + c^2 + d^2 - ij \cdot 2(ad-bc)
\end{aligned}
$$

$$
\frac{1}{2} \left(z * \bar{z}_{ij} + \bar{z}_i * \bar{z}_j \right) = a^2 + b^2 + c^2 + d^2
$$

$$
\frac{1}{2} \left(z * \bar{z}_i + \bar{z}_{ij} * \bar{z}_j \right) = a^2 + b^2 - c^2 - d^2
$$

$$
\frac{1}{2} \left(z * \bar{z}_j + \bar{z}_{ij} * \bar{z}_i \right) = a^2 - b^2 + c^2 - d^2
$$

6 Dot product $< u, v >$ $(+, +, +, +)$

$$
\begin{aligned}
z &= a + ib + jc + ijd \\
\bar{z}_{ij} &= a - ib - jc + ijd \\
\bar{z}_i &= a - ib + jc - ijd \\
\bar{z}_j &= a + ib - jc - ijd
\end{aligned}
$$

$$
\begin{aligned}
u &= a_1 + ib_1 + jc_1 + ijd_1 \\
\bar{u}_{ij} &= a_1 - ib_1 - jc_1 + ijd_1 \\
\bar{u}_i &= a_1 - ib_1 + jc_1 - ijd_1 \\
\bar{u}_j &= a_1 + ib_1 - jc_1 - ijd_1
\end{aligned}
$$

$$
\begin{aligned}
v &= a_2 + ib_2 + jc_2 + ijd_2 \\
\bar{v}_{ij} &= a_2 - ib_2 - jc_2 + ijd_2 \\
\bar{v}_i &= a_2 - ib_2 + jc_2 - ijd_2 \\
\bar{v}_j &= a_2 + ib_2 - jc_2 - ijd_2
\end{aligned}
$$

$$
< u, v > \overset{\text{def}}{=} \frac{1}{4} \left(\left(u * \bar{v}_{ij} + v * \bar{u}_{ij} \right) + \left(\bar{u}_i * \bar{v}_j + \bar{v}_i * \bar{u}_j \right) \right)
$$

$$
< u, v > = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2
$$

$$
< z, z > \overset{\text{def}}{=} \frac{1}{2} \left(z * \bar{z}_{ij} + \bar{z}_i * \bar{z}_j \right)
$$

$$
< z, z > = a^2 + b^2 + c^2 + d^2
$$

7 Conjugate $< bcd >$

7.1

$$
\begin{aligned}
z &= a + ib + jc + ijd \\
\bar{z}_d &= a + ib + jc - ijd \\
\bar{z}_c &= a + ib - jc + ijd \\
\bar{z}_{cd} &= a + ib - jc - ijd = \bar{z}_j = (a+ib) - j(c+id) \\
\bar{z}_b &= a - ib + jc + ijd \\
\bar{z}_{bd} &= a - ib + jc - ijd = \bar{z}_i = (a-ib) + j(c-id) \\
\bar{z}_{bc} &= a - ib - jc + ijd = \bar{z}_{ij} = (a-ib) - j(c-id) \\
\bar{z}_{bcd} &= a - ib - jc - ijd
\end{aligned}
$$

7.2

$$
\begin{aligned}
z \ \ * \ \bar{z}_d &= a^2 - b^2 - c^2 - d^2 + i\cdot(+2ab) + j\cdot(+2ac) + ij\cdot(+2bc) \\
\bar{z}_{bc} * \bar{z}_{bcd} &= a^2 - b^2 - c^2 - d^2 + i\cdot(-2ab) + j\cdot(-2ac) + ij\cdot(+2bc) \\
\bar{z}_{bd} * \bar{z}_b &= a^2 - b^2 - c^2 - d^2 + i\cdot(-2ab) + j\cdot(+2ac) + ij\cdot(-2bc) \\
\bar{z}_{cd} * \bar{z}_c &= a^2 - b^2 - c^2 - d^2 + i\cdot(+2ab) + j\cdot(-2ac) + ij\cdot(-2bc)
\end{aligned}
$$

$$
\frac{1}{4}\left(z*\bar{z}_d + \bar{z}_{bc}*\bar{z}_{bcd} + \bar{z}_{bd}*\bar{z}_b + \bar{z}_{cd}*\bar{z}_c \right) = (a^2 - b^2 - c^2 - d^2)
$$

7.3

$$
\begin{aligned}
z \ \ * \ \bar{z}_{bcd} &= a^2 + b^2 + c^2 - d^2 + i\cdot(+2cd) + j\cdot(+2bd) + ij\cdot(-2bc) \\
\bar{z}_{bc} * \bar{z}_d &= a^2 + b^2 + c^2 - d^2 + i\cdot(-2cd) + j\cdot(-2bd) + ij\cdot(-2bc) \\
\bar{z}_{bd} * \bar{z}_c &= a^2 + b^2 + c^2 - d^2 + i\cdot(-2cd) + j\cdot(+2bd) + ij\cdot(+2bc) \\
\bar{z}_{cd} * \bar{z}_b &= a^2 + b^2 + c^2 - d^2 + i\cdot(+2cd) + j\cdot(-2bd) + ij\cdot(+2bc)
\end{aligned}
$$

$$
\frac{1}{4}\left(z*\bar{z}_{bcd} + \bar{z}_{bc}*\bar{z}_d + \bar{z}_{bd}*\bar{z}_c + \bar{z}_{cd}*\bar{z}_b \right) = (a^2 + b^2 + c^2 - d^2)
$$

7.4

$$
\begin{aligned}
z &= a \ \ + ib \ \ + jc \ \ + ijd \\
u &= a_1 + ib_1 + jc_1 + ijd_1 \\
v &= a_2 + ib_2 + jc_2 + ijd_2
\end{aligned}
$$

$$
\frac{1}{4}\left(z*\bar{z}_d + \bar{z}_{bc}*\bar{z}_{bcd} + \bar{z}_{bd}*\bar{z}_b + \bar{z}_{cd}*\bar{z}_c \right) = (a^2 - b^2 - c^2 - d^2)
$$

$$
\frac{1}{8}\left(\left(u*\bar{v}_d + v*\bar{u}_d\right) + \left(\bar{u}_{bc}*\bar{v}_{bcd} + \bar{v}_{bc}*\bar{u}_{bcd}\right) + \left(\bar{u}_{bd}*\bar{v}_b + \bar{v}_{bd}*\bar{u}_b\right) + \left(\bar{u}_{cd}*\bar{v}_c + \bar{v}_{cd}*\bar{u}_c\right) \right) = (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2)
$$

$$
\frac{1}{4}\left(z*\bar{z}_{bcd} + \bar{z}_{bc}*\bar{z}_d + \bar{z}_{bd}*\bar{z}_c + \bar{z}_{cd}*\bar{z}_b \right) = (a^2 + b^2 + c^2 - d^2)
$$

$$
\frac{1}{8}\left(\left(u*\bar{v}_{bcd} + v*\bar{u}_{bcd}\right) + \left(\bar{u}_{bc}*\bar{v}_d + \bar{v}_{bc}*\bar{u}_d\right) + \left(\bar{u}_{bd}*\bar{v}_c + \bar{v}_{bd}*\bar{u}_c\right) + \left(\bar{u}_{cd}*\bar{v}_b + \bar{v}_{cd}*\bar{u}_b\right) \right) = (a_1 a_2 + b_1 b_2 + c_1 c_2 - d_1 d_2)
$$

8 Dot product $< u, v >$ $(+, -, -, -)$

$z = a \ + ib \ + jc \ + ijd$
$u = a_1 + ib_1 + jc_1 + ijd_1$
$v = a_2 + ib_2 + jc_2 + ijd_2$

$< u, v > \overset{\text{def}}{=} \dfrac{1}{8} \left(\left(u * \bar{v}_d + v * \bar{u}_d \right) + \left(\bar{u}_{bc} * \bar{v}_{bcd} + \bar{v}_{bc} * \bar{u}_{bcd} \right) + \left(\bar{u}_{bd} * \bar{v}_b + \bar{v}_{bd} * \bar{u}_b \right) + \left(\bar{u}_{cd} * \bar{v}_c + \bar{v}_{cd} * \bar{u}_c \right) \right)$

$< u, v > \ = \ a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2$

$< z, z > \overset{\text{def}}{=} \dfrac{1}{4} \left(z * \bar{z}_d + \bar{z}_{bc} * \bar{z}_{bcd} + \bar{z}_{bd} * \bar{z}_b + \bar{z}_{cd} * \bar{z}_c \right)$

$< z, z > \ = \ a^2 - b^2 - c^2 - d^2$

9 Dot product $< u, v >$ $(+, +, +, -)$

$z = a \ + ib \ + jc \ + ijd$
$u = a_1 + ib_1 + jc_1 + ijd_1$
$v = a_2 + ib_2 + jc_2 + ijd_2$

$< u, v > \overset{\text{def}}{=} \dfrac{1}{8} \left(\left(u * \bar{v}_{bcd} + v * \bar{u}_{bcd} \right) + \left(\bar{u}_{bc} * \bar{v}_d + \bar{v}_{bc} * \bar{u}_d \right) + \left(\bar{u}_{bd} * \bar{v}_c + \bar{v}_{bd} * \bar{u}_c \right) + \left(\bar{u}_{cd} * \bar{v}_b + \bar{v}_{cd} * \bar{u}_b \right) \right)$

$< u, v > \ = \ a_1 a_2 + b_1 b_2 + c_1 c_2 - d_1 d_2$

$< z, z > \overset{\text{def}}{=} \dfrac{1}{4} \left(z * \bar{z}_{bcd} + \bar{z}_{bc} * \bar{z}_d + \bar{z}_{bd} * \bar{z}_c + \bar{z}_{cd} * \bar{z}_b \right)$

$< z, z > \ = \ a^2 + b^2 + c^2 - d^2$

10 Multiplication table

10.1

$$z \quad = a + ib + jc + ijd$$
$$\bar{z}_d \quad = a + ib + jc - ijd$$
$$\bar{z}_c \quad = a + ib - jc + ijd$$
$$\bar{z}_{cd} = a + ib - jc - ijd = \bar{z}_j \quad = (a+ib) - j(c+id)$$
$$\bar{z}_b \quad = a - ib + jc + ijd$$
$$\bar{z}_{bd} = a - ib + jc - ijd = \bar{z}_i \quad = (a-ib) + j(c-id)$$
$$\bar{z}_{bc} = a - ib - jc + ijd = \bar{z}_{ij} = (a-ib) - j(c-id)$$
$$\bar{z}_{bcd} = a - ib - jc - ijd$$

10.2

$$z \quad * z \quad = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab-cd)) + j \cdot (+2(ac-bd)) + ij \cdot (+2(ad+bc))$$
$$z \quad * \bar{z}_d \quad = a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (+2ac) + ij \cdot (+2bc)$$
$$z \quad * \bar{z}_c \quad = a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (-2bd) + ij \cdot (+2ad)$$
$$z \quad * \bar{z}_{cd} \quad = a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab+cd))$$
$$z \quad * \bar{z}_b \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (+2ac) + ij \cdot (+2ad)$$
$$z \quad * \bar{z}_{bd} \quad = a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac+bd))$$
$$z \quad * \bar{z}_{bc} \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad-bc))$$
$$z \quad * \bar{z}_{bcd} = a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (+2bd) + ij \cdot (-2bc)$$

$$\bar{z}_d \quad * z \quad = a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (+2ac) + ij \cdot (+2bc)$$
$$\bar{z}_d \quad * \bar{z}_d \quad = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab+cd)) + j \cdot (+2(ac+bd)) + ij \cdot (-2(ad-bc))$$
$$\bar{z}_d \quad * \bar{z}_c \quad = a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab-cd))$$
$$\bar{z}_d \quad * \bar{z}_{cd} \quad = a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (+2bd) + ij \cdot (-2ad)$$
$$\bar{z}_d \quad * \bar{z}_b \quad = a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac-bd))$$
$$\bar{z}_d \quad * \bar{z}_{bd} \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (+2ac) + ij \cdot (-2ad)$$
$$\bar{z}_d \quad * \bar{z}_{bc} \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (-2bd) + ij \cdot (-2bc)$$
$$\bar{z}_d \quad * \bar{z}_{bcd} = a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad+bc))$$

$$\bar{z}_c \quad * z \quad = a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (-2bd) + ij \cdot (+2ad)$$
$$\bar{z}_c \quad * \bar{z}_d \quad = a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab-cd))$$
$$\bar{z}_c \quad * \bar{z}_c \quad = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab+cd)) + j \cdot (-2(ac+bd)) + ij \cdot (+2(ad-bc))$$
$$\bar{z}_c \quad * \bar{z}_{cd} \quad = a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (-2ac) + ij \cdot (-2bc)$$
$$\bar{z}_c \quad * \bar{z}_b \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad+bc))$$
$$\bar{z}_c \quad * \bar{z}_{bd} \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (+2bd) + ij \cdot (+2bc)$$
$$\bar{z}_c \quad * \bar{z}_{bc} \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (-2ac) + ij \cdot (+2ad)$$
$$\bar{z}_c \quad * \bar{z}_{bcd} = a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac-bd))$$

$$\bar{z}_{cd} * z \quad = a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab+cd))$$
$$\bar{z}_{cd} * \bar{z}_d \quad = a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (+2bd) + ij \cdot (-2ad)$$
$$\bar{z}_{cd} * \bar{z}_c \quad = a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (-2ac) + ij \cdot (-2bc)$$
$$\bar{z}_{cd} * \bar{z}_{cd} = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab-cd)) + j \cdot (-2(ac-bd)) + ij \cdot (-2(ad+bc))$$
$$\bar{z}_{cd} * \bar{z}_b \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (-2bd) + ij \cdot (+2bc)$$
$$\bar{z}_{cd} * \bar{z}_{bd} \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad-bc))$$
$$\bar{z}_{cd} * \bar{z}_{bc} \quad = a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac+bd))$$
$$\bar{z}_{cd} * \bar{z}_{bcd} = a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (-2ac) + ij \cdot (-2ad)$$

$$\bar{z}_b \quad * z \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (+2ac) + ij \cdot (+2ad)$$
$$\bar{z}_b \quad * \bar{z}_d \quad = a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac-bd))$$
$$\bar{z}_b \quad * \bar{z}_c \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad+bc))$$
$$\bar{z}_b \quad * \bar{z}_{cd} \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (-2bd) + ij \cdot (+2bc)$$
$$\bar{z}_b \quad * \bar{z}_b \quad = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab+cd)) + j \cdot (+2(ac+bd)) + ij \cdot (+2(ad-bc))$$
$$\bar{z}_b \quad * \bar{z}_{bd} \quad = a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (+2ac) + ij \cdot (-2bc)$$
$$\bar{z}_b \quad * \bar{z}_{bc} \quad = a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (+2bd) + ij \cdot (+2ad)$$
$$\bar{z}_b \quad * \bar{z}_{bcd} = a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab-cd))$$

$$\bar{z}_{bd} * z \quad = a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac + bd))$$
$$\bar{z}_{bd} * \bar{z}_d \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (+2ac) + ij \cdot (-2ad)$$
$$\bar{z}_{bd} * \bar{z}_c \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (+2bd) + ij \cdot (+2bc)$$
$$\bar{z}_{bd} * \bar{z}_{cd} = a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad - bc))$$
$$\bar{z}_{bd} * \bar{z}_b \quad = a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (+2ac) + ij \cdot (-2bc)$$
$$\bar{z}_{bd} * \bar{z}_{bd} = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab - cd)) + j \cdot (+2(ac - bd)) + ij \cdot (-2(ad + bc))$$
$$\bar{z}_{bd} * \bar{z}_{bc} = a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab + cd))$$
$$\bar{z}_{bd} * \bar{z}_{bcd} = a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (-2bd) + ij \cdot (-2ad)$$

$$\bar{z}_{bc} * z \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad - bc))$$
$$\bar{z}_{bc} * \bar{z}_d \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (-2bd) + ij \cdot (-2bc)$$
$$\bar{z}_{bc} * \bar{z}_c \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (-2ac) + ij \cdot (+2ad)$$
$$\bar{z}_{bc} * \bar{z}_{cd} = a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac + bd))$$
$$\bar{z}_{bc} * \bar{z}_b \quad = a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (+2bd) + ij \cdot (+2ad)$$
$$\bar{z}_{bc} * \bar{z}_{bd} = a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab + cd))$$
$$\bar{z}_{bc} * \bar{z}_{bc} = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab - cd)) + j \cdot (-2(ac - bd)) + ij \cdot (+2(ad + bc))$$
$$\bar{z}_{bc} * \bar{z}_{bcd} = a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (-2ac) + ij \cdot (+2bc)$$

$$\bar{z}_{bcd} * z \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (+2bd) + ij \cdot (-2bc)$$
$$\bar{z}_{bcd} * \bar{z}_d \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad + bc))$$
$$\bar{z}_{bcd} * \bar{z}_c \quad = a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac - bd))$$
$$\bar{z}_{bcd} * \bar{z}_{cd} = a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (-2ac) + ij \cdot (-2ad)$$
$$\bar{z}_{bcd} * \bar{z}_b \quad = a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab - cd))$$
$$\bar{z}_{bcd} * \bar{z}_{bd} = a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (-2bd) + ij \cdot (-2ad)$$
$$\bar{z}_{bcd} * \bar{z}_{bc} = a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (-2ac) + ij \cdot (+2bc)$$
$$\bar{z}_{bcd} * \bar{z}_{bcd} = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab + cd)) + j \cdot (-2(ac + bd)) + ij \cdot (-2(ad - bc))$$

10.3

$$z \quad * \bar{z}_{bc} \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad - bc))$$
$$\bar{z}_d \quad * \bar{z}_{bcd} = a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad + bc))$$
$$\bar{z}_c \quad * \bar{z}_b \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad + bc))$$
$$\bar{z}_{cd} \quad * \bar{z}_{bd} \quad = a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad - bc))$$
$$\bar{z}_b \quad * \bar{z}_c \quad \doteq a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad + bc))$$
$$\bar{z}_{bd} \quad * \bar{z}_{cd} \quad \doteq a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad - bc))$$
$$\bar{z}_{bc} \quad * z \quad \doteq a^2 + b^2 + c^2 + d^2 + ij \cdot (+2(ad - bc))$$
$$\bar{z}_{bcd} * \bar{z}_d \quad \doteq a^2 + b^2 + c^2 + d^2 + ij \cdot (-2(ad + bc))$$

$$z \quad * \bar{z}_{bcd} = a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (+2bd) + ij \cdot (-2bc)$$
$$\bar{z}_d \quad * \bar{z}_{bc} \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (-2bd) + ij \cdot (-2bc)$$
$$\bar{z}_c \quad * \bar{z}_{bd} \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (+2bd) + ij \cdot (+2bc)$$
$$\bar{z}_{cd} \quad * \bar{z}_b \quad = a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (-2bd) + ij \cdot (+2bc)$$
$$\bar{z}_b \quad * \bar{z}_{cd} \quad \doteq a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (-2bd) + ij \cdot (+2bc)$$
$$\bar{z}_{bd} \quad * \bar{z}_c \quad \doteq a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (+2bd) + ij \cdot (+2bc)$$
$$\bar{z}_{bc} \quad * \bar{z}_d \quad \doteq a^2 + b^2 + c^2 - d^2 + i \cdot (-2cd) + j \cdot (-2bd) + ij \cdot (-2bc)$$
$$\bar{z}_{bcd} * z \quad \doteq a^2 + b^2 + c^2 - d^2 + i \cdot (+2cd) + j \cdot (+2bd) + ij \cdot (-2bc)$$

$$z \quad * \bar{z}_b \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (+2ac) + ij \cdot (+2ad)$$
$$\bar{z}_d \quad * \bar{z}_{bd} \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (+2ac) + ij \cdot (-2ad)$$
$$\bar{z}_c \quad * \bar{z}_{bc} \quad = a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (-2ac) + ij \cdot (+2ad)$$
$$\bar{z}_{cd} \quad * \bar{z}_{bcd} = a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (-2ac) + ij \cdot (-2ad)$$
$$\bar{z}_b \quad * z \quad \doteq a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (+2ac) + ij \cdot (+2ad)$$
$$\bar{z}_{bd} \quad * \bar{z}_d \quad \doteq a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (+2ac) + ij \cdot (-2ad)$$
$$\bar{z}_{bc} \quad * \bar{z}_c \quad \doteq a^2 + b^2 - c^2 + d^2 + i \cdot (+2cd) + j \cdot (-2ac) + ij \cdot (+2ad)$$
$$\bar{z}_{bcd} * \bar{z}_{cd} \quad \doteq a^2 + b^2 - c^2 + d^2 + i \cdot (-2cd) + j \cdot (-2ac) + ij \cdot (-2ad)$$

$$z \quad * \bar{z}_{bd} = a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac + bd))$$
$$\bar{z}_d \quad * \bar{z}_b = a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac - bd))$$
$$\bar{z}_c \quad * \bar{z}_{bcd} = a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac - bd))$$
$$\bar{z}_{cd} * \bar{z}_{bc} = a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac + bd))$$
$$\bar{z}_b \quad * \bar{z}_d \doteq a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac - bd))$$
$$\bar{z}_{bd} * z \doteq a^2 + b^2 - c^2 - d^2 + j \cdot (+2(ac + bd))$$
$$\bar{z}_{bc} * \bar{z}_{cd} \doteq a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac + bd))$$
$$\bar{z}_{bcd} * \bar{z}_c \doteq a^2 + b^2 - c^2 - d^2 + j \cdot (-2(ac - bd))$$

$$z \quad * \bar{z}_c = a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (-2bd) + ij \cdot (+2ad)$$
$$\bar{z}_d \quad * \bar{z}_{cd} = a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (+2bd) + ij \cdot (-2ad)$$
$$\bar{z}_c \quad * z \doteq a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (-2bd) + ij \cdot (+2ad)$$
$$\bar{z}_{cd} * \bar{z}_d \doteq a^2 - b^2 + c^2 + d^2 + i \cdot (+2ab) + j \cdot (+2bd) + ij \cdot (-2ad)$$
$$\bar{z}_b \quad * \bar{z}_{bc} = a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (+2bd) + ij \cdot (+2ad)$$
$$\bar{z}_{bd} * \bar{z}_{bcd} = a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (-2bd) + ij \cdot (-2ad)$$
$$\bar{z}_{bc} * \bar{z}_b \doteq a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (+2bd) + ij \cdot (+2ad)$$
$$\bar{z}_{bcd} * \bar{z}_{bd} \doteq a^2 - b^2 + c^2 + d^2 + i \cdot (-2ab) + j \cdot (-2bd) + ij \cdot (-2ad)$$

$$z \quad * \bar{z}_{cd} = a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab + cd))$$
$$\bar{z}_d \quad * \bar{z}_c = a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab - cd))$$
$$\bar{z}_c \quad * \bar{z}_d \doteq a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab - cd))$$
$$\bar{z}_{cd} * z \doteq a^2 - b^2 + c^2 - d^2 + i \cdot (+2(ab + cd))$$
$$\bar{z}_b \quad * \bar{z}_{bcd} = a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab - cd))$$
$$\bar{z}_{bd} * \bar{z}_{bc} = a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab + cd))$$
$$\bar{z}_{bc} * \bar{z}_{bd} \doteq a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab + cd))$$
$$\bar{z}_{bcd} * \bar{z}_b \doteq a^2 - b^2 + c^2 - d^2 + i \cdot (-2(ab - cd))$$

$$z \quad * z = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab - cd)) + j \cdot (+2(ac - bd)) + ij \cdot (+2(ad + bc))$$
$$\bar{z}_d \quad * \bar{z}_d = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab + cd)) + j \cdot (+2(ac + bd)) + ij \cdot (-2(ad - bc))$$
$$\bar{z}_c \quad * \bar{z}_c = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab + cd)) + j \cdot (-2(ac + bd)) + ij \cdot (+2(ad - bc))$$
$$\bar{z}_{cd} * \bar{z}_{cd} = a^2 - b^2 - c^2 + d^2 + i \cdot (+2(ab - cd)) + j \cdot (-2(ac - bd)) + ij \cdot (-2(ad + bc))$$
$$\bar{z}_b \quad * \bar{z}_b = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab + cd)) + j \cdot (+2(ac + bd)) + ij \cdot (+2(ad - bc))$$
$$\bar{z}_{bd} * \bar{z}_{bd} = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab - cd)) + j \cdot (+2(ac - bd)) + ij \cdot (-2(ad + bc))$$
$$\bar{z}_{bc} * \bar{z}_{bc} = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab - cd)) + j \cdot (-2(ac - bd)) + ij \cdot (+2(ad + bc))$$
$$\bar{z}_{bcd} * \bar{z}_{bcd} = a^2 - b^2 - c^2 + d^2 + i \cdot (-2(ab + cd)) + j \cdot (-2(ac + bd)) + ij \cdot (-2(ad - bc))$$

$$z \quad * \bar{z}_d = a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (+2ac) + ij \cdot (+2bc)$$
$$\bar{z}_d \quad * z \doteq a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (+2ac) + ij \cdot (+2bc)$$
$$\bar{z}_c \quad * \bar{z}_{cd} = a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (-2ac) + ij \cdot (-2bc)$$
$$\bar{z}_{cd} * \bar{z}_c \doteq a^2 - b^2 - c^2 - d^2 + i \cdot (+2ab) + j \cdot (-2ac) + ij \cdot (-2bc)$$
$$\bar{z}_b \quad * \bar{z}_{bd} = a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (+2ac) + ij \cdot (-2bc)$$
$$\bar{z}_{bd} * \bar{z}_b \doteq a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (+2ac) + ij \cdot (-2bc)$$
$$\bar{z}_{bc} * \bar{z}_{bcd} = a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (-2ac) + ij \cdot (+2bc)$$
$$\bar{z}_{bcd} * \bar{z}_{bc} \doteq a^2 - b^2 - c^2 - d^2 + i \cdot (-2ab) + j \cdot (-2ac) + ij \cdot (+2bc)$$

10.4

$$a^2 + b^2 + c^2 + d^2 = 1/2 \cdot (z * \bar{z}_{bc} + \bar{z}_{bd} * \bar{z}_{cd})$$
$$a^2 + b^2 + c^2 + d^2 = 1/2 \cdot (\bar{z}_b * \bar{z}_c + \bar{z}_d * \bar{z}_{bcd})$$
$$a^2 + b^2 + c^2 - d^2 = 1/4 \cdot (z * \bar{z}_{bcd} + \bar{z}_b * \bar{z}_{cd} + \bar{z}_c * \bar{z}_{bd} + \bar{z}_d * \bar{z}_{bc})$$
$$a^2 + b^2 - c^2 + d^2 = 1/4 \cdot (z * \bar{z}_b + \bar{z}_c * \bar{z}_{bc} + \bar{z}_d * \bar{z}_{bd} + \bar{z}_{cd} * \bar{z}_{bcd})$$
$$a^2 + b^2 - c^2 - d^2 = 1/2 \cdot (z * \bar{z}_{bd} + \bar{z}_{bc} * \bar{z}_{cd})$$
$$a^2 + b^2 - c^2 - d^2 = 1/2 \cdot (\bar{z}_b * \bar{z}_d + \bar{z}_c * \bar{z}_{bcd})$$
$$a^2 - b^2 + c^2 + d^2 = 1/4 \cdot (z * \bar{z}_c + \bar{z}_b * \bar{z}_{bc} + \bar{z}_d * \bar{z}_{cd} + \bar{z}_{bd} * \bar{z}_{bcd})$$
$$a^2 - b^2 + c^2 - d^2 = 1/2 \cdot (z * \bar{z}_{cd} + \bar{z}_{bc} * \bar{z}_{bd})$$
$$a^2 - b^2 + c^2 - d^2 = 1/2 \cdot (\bar{z}_b * \bar{z}_{bcd} + \bar{z}_c * \bar{z}_d)$$
$$a^2 - b^2 - c^2 + d^2 = 1/4 \cdot (z * z + \bar{z}_{bc} * \bar{z}_{bc} + \bar{z}_{bd} * \bar{z}_{bd} + \bar{z}_{cd} * \bar{z}_{cd})$$
$$a^2 - b^2 - c^2 + d^2 = 1/4 \cdot (\bar{z}_b * \bar{z}_b + \bar{z}_c * \bar{z}_c + \bar{z}_d * \bar{z}_d + \bar{z}_{bcd} * \bar{z}_{bcd})$$
$$a^2 - b^2 - c^2 - d^2 = 1/4 \cdot (z * \bar{z}_d + \bar{z}_b * \bar{z}_{bd} + \bar{z}_c * \bar{z}_{cd} + \bar{z}_{bc} * \bar{z}_{bcd})$$

11 Exponential form

$$z = (a + ib) + j \cdot (c + id) = r * e^{i\phi} + j \cdot \rho * e^{i\psi}$$
$$z = |z| * e^{i\alpha + j\beta + ij\gamma} = R * e^{i\alpha + j\beta + ij\gamma}$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$
$$e^{j\beta} = \cos\beta + j\sin\beta$$
$$e^{ij\gamma} = \cosh\gamma + ij\sinh\gamma$$

$$\left|e^{i\phi}\right| = \left|e^{j\phi}\right| = \left|e^{ij\phi}\right| = 1$$

$$i = e^{i\pi/2}$$
$$j = e^{j\pi/2}$$
$$ij = e^{(i+j)\pi/2}$$

$$\sinh\gamma + ij\cosh\gamma = ij \cdot (\cosh\gamma + ij\sinh\gamma) = ij \cdot e^{ij\gamma}$$
$$\sinh\gamma + ij\cosh\gamma = e^{(i+j)\pi/2 + ij\gamma}$$

$$z = R * (\cos\alpha + i\sin\alpha)(\cos\beta + j\sin\beta)(\cosh\gamma + ij\sinh\gamma)$$
$$z = R * (\cos\alpha\cos\beta + i\sin\alpha\cos\beta + j\cos\alpha\sin\beta + ij\sin\alpha\sin\beta)(\cosh\gamma + ij\sinh\gamma)$$

$$a = R * (\cosh\gamma\cos\alpha\cos\beta + \sinh\gamma\sin\alpha\sin\beta)$$
$$b = R * (\cosh\gamma\sin\alpha\cos\beta - \sinh\gamma\cos\alpha\sin\beta)$$
$$c = R * (\cosh\gamma\cos\alpha\sin\beta - \sinh\gamma\sin\alpha\cos\beta)$$
$$d = R * (\cosh\gamma\sin\alpha\sin\beta + \sinh\gamma\cos\alpha\cos\beta)$$

$$\begin{cases} a + d = R * e^{+\gamma} * \cos(\alpha - \beta) \\ b - c = R * e^{+\gamma} * \sin(\alpha - \beta) \\ a - d = R * e^{-\gamma} * \cos(\alpha + \beta) \\ b + c = R * e^{-\gamma} * \sin(\alpha + \beta) \end{cases}$$

$$\begin{cases} (a+d)^2 + (b-c)^2 = R^2 * e^{2\gamma} \\ (a-d)^2 + (b+c)^2 = R^2 * e^{-2\gamma} \end{cases}$$

$$\begin{cases} a^2 + b^2 + c^2 + d^2 = R^2 * \cosh 2\gamma \\ 2(ad - bc) = R^2 * \sinh 2\gamma \end{cases}$$

$$\begin{cases} \tan(\alpha - \beta) = \dfrac{b - c}{a + d} = \arg(a+d, b-c) \\[2ex] \tan(\alpha + \beta) = \dfrac{b + c}{a - d} = \arg(a-d, b+c) \end{cases}$$

$$R = \sqrt[4]{\left((a+d)^2 + (b-c)^2\right) * \left((a-d)^2 + (b+c)^2\right)}$$

$$R = \sqrt[4]{\left(r^2 + \rho^2\right)^2 - 4r^2\rho^2 * \sin^2(\psi - \phi)}$$

$$R = \sqrt[4]{r^4 + \rho^4 + 2r^2\rho^2 * \cos 2(\psi - \phi)}$$

$$\gamma = \frac{1}{4} \ln \frac{(a+d)^2 + (b-c)^2}{(a-d)^2 + (b+c)^2}$$

$$\gamma = \frac{1}{4} \ln \frac{r^2 + \rho^2 + 2r\rho * \sin(\psi - \phi)}{r^2 + \rho^2 - 2r\rho * \sin(\psi - \phi)}$$

$$\alpha = \frac{1}{2}\left[\arg(a+d, b-c) + \arg(a-d, b+c)\right]$$
$$\beta = \frac{1}{2}\left[\arg(a-d, b+c) - \arg(a+d, b-c)\right]$$
$$\arg(x, y) = \arctan(y, x) = \arctan(y : x) = \arctan(y/x)$$

11.1

$$R^2 = r^2 + \rho^2 - 2r\rho \cdot \cos\theta, \quad \cos\theta = ?$$

$$\cos\theta = \frac{1}{2r\rho} * (r^2 + \rho^2 - R^2)$$

$$\cos\theta = \frac{1}{2r\rho} * \left(r^2 + \rho^2 - \sqrt{(r^2 + \rho^2)^2 - 4r^2\rho^2 * \sin^2(\psi - \phi)}\right)$$

$$\cos\theta = \frac{r^2 + \rho^2}{2r\rho} - \sqrt{\left(\frac{r^2 + \rho^2}{2r\rho}\right)^2 - \sin^2(\psi - \phi)}$$

$$\frac{r^2 + \rho^2}{2r\rho} - 1 = \frac{r^2 + \rho^2 - 2r\rho}{2r\rho} = \frac{(r - \rho)^2}{2r\rho} \geq 0, \quad \frac{r^2 + \rho^2}{2r\rho} \geq 1$$

$$(\cos\theta)_{min} = 0$$

$$(\cos\theta)_{max} = \frac{r^2 + \rho^2}{2r\rho} - \sqrt{\left(\frac{r^2 + \rho^2}{2r\rho}\right)^2 - 1} = \frac{1}{2r\rho} * \left(r^2 + \rho^2 - (r + \rho) \cdot |r - \rho|\right)$$

$$(\cos\theta)_{max} = \frac{\rho}{r}, \ r > \rho$$

$$(\cos\theta)_{max} = \frac{r}{\rho}, \ \rho > r$$

11.2

$$\begin{cases} e^{+\gamma} = \sqrt[4]{\dfrac{(a + d)^2 + (b - c)^2}{(a - d)^2 + (b + c)^2}} \\[4ex] e^{-\gamma} = \sqrt[4]{\dfrac{(a - d)^2 + (b + c)^2}{(a + d)^2 + (b - c)^2}} \end{cases}$$

$$\begin{cases} R * e^{+\gamma} = \sqrt{(a + d)^2 + (b - c)^2} \\[2ex] R * e^{-\gamma} = \sqrt{(a - d)^2 + (b + c)^2} \end{cases}$$

$$\begin{cases} \alpha - \beta = \arg(a + d, b - c) \\ \alpha + \beta = \arg(a - d, b + c) \end{cases}$$

11.3

$$\gamma = 0 \Rightarrow$$

$$a \cdot d = b \cdot c$$

$$R^2 = a^2 + b^2 + c^2 + d^2$$
$$R^2 = (a + d)^2 + (b - c)^2$$
$$R^2 = (a - d)^2 + (b + c)^2$$

11.4

$$\phi = \psi \Rightarrow$$

$$z = (a + ib) \cdot (c + jd) = ac + i \cdot bc + j \cdot ad + ij \cdot bd = \underline{a} + i \cdot \underline{b} + j \cdot \underline{c} + ij \cdot \underline{d}$$

$$\underline{a} = a \cdot c$$
$$\underline{b} = b \cdot c$$
$$\underline{c} = a \cdot d$$
$$\underline{d} = b \cdot d$$

$$\underline{a} \cdot \underline{d} = \underline{b} \cdot \underline{c} = a \cdot b \cdot c \cdot d$$

$$R^2 = \underline{a}^2 + \underline{b}^2 + \underline{c}^2 + \underline{d}^2$$
$$\gamma = 0$$

11.5 R=0

$$z = (a + ib) + j(c + id) = a + ib + jc + ijd = R * e^{i\alpha + j\beta + ij\gamma}$$

$$R = \sqrt[4]{\left((a + d)^2 + (b - c)^2\right) * \left((a - d)^2 + (b + c)^2\right)}$$

$R = 0$
$z = (0) * e^{j\theta}$

1) $a = d \neq 0 \;\wedge\; b = -c \neq 0$
   $z = a(1 + ij) + b(i - j) = a + ib - jb + ija = (a + ib) + j(-b + ia) = (a + ib) + j(i(a + ib))$
   $\theta = \arctan(+i)$

2) $a = -d \neq 0 \;\wedge\; b = c \neq 0$
   $z = a(1 - ij) + b(i + j) = a + ib + jb - ija = (a + ib) + j(b - ia) = (a + ib) + j(-i(a + ib))$
   $\theta = \arctan(-i)$

3) $a = d = 0 \;\wedge\; b = c \neq 0$
   $z = b(i + j) = ib + jb = (ib) + j(b)$
   $\theta = \arctan(-i)$

4) $a = d = 0 \;\wedge\; b = -c \neq 0$
   $z = b(i - j) = ib - jb = (ib) + j(-b)$
   $\theta = \arctan(+i)$

5) $b = c = 0 \;\wedge\; a = d \neq 0$
   $z = a(1 + ij) = a + ija = (a) + j(ia)$
   $\theta = \arctan(+i)$

6) $b = c = 0 \;\wedge\; a = -d \neq 0$
   $z = a(1 - ij) = a - ija = (a) + j(-ia)$
   $\theta = \arctan(-i)$

7) $a = b = c = d = 0$
   $z = 0$

$\arctan(+i) = +i \cdot \infty$
$\arctan(-i) = -i \cdot \infty$

$$
\begin{array}{lcl}
(1 + ij) * (1 + ij) = 1 + ij + ij + 1 & = & 2 \cdot (1 + ij) \\
(1 + ij) * (i - j) = i - j - j + i & = & 2 \cdot (i - j) \\
(1 + ij) * (1 - ij) = 1 - ij + ij - 1 & = & 0 \\
(1 + ij) * (i + j) = i + j - j - i & = & 0 \\
\\
(i - j) * (1 + ij) = i - j - j + i & = & 2 \cdot (i - j) \\
(i - j) * (i - j) = -1 - ij - ij - 1 & = & -2 \cdot (1 + ij) \\
(i - j) * (1 - ij) = i + j - j - i & = & 0 \\
(i - j) * (i + j) = -1 + ij - ij + 1 & = & 0 \\
\\
(1 - ij) * (1 + ij) = 1 + ij - ij - 1 & = & 0 \\
(1 - ij) * (i - j) = i - j + j - i & = & 0 \\
(1 - ij) * (1 - ij) = 1 - ij - ij + 1 & = & 2 \cdot (1 - ij) \\
(1 - ij) * (i + j) = i + j + j + i & = & 2 \cdot (i + j) \\
\\
(i + j) * (1 + ij) = i - j + j - i & = & 0 \\
(i + j) * (i - j) = -1 - ij_i j + 1 & = & 0 \\
(i + j) * (1 - ij) = i + j + j + i & = & 2 \cdot (i + j) \\
(i + j) * (i + j) = -1 + ij + ij - 1 & = & -2 \cdot (1 - ij)
\end{array}
$$

12 Conjugate ±

12.1 e+

$$z = e^{i\alpha + j\beta + ij\gamma} = (a + ib) + j(c + id)$$

$$z = (\cos\alpha + i\sin\alpha)(\cos\beta + j\sin\beta)(\cosh\gamma + ij\sinh\gamma)$$
$$z = (\cos\alpha\cos\beta + i\sin\alpha\cos\beta + j\cos\alpha\sin\beta + ij\sin\alpha\sin\beta)(\cosh\gamma + ij\sinh\gamma)$$

$$a = \cosh\gamma\cos\alpha\cos\beta + \sinh\gamma\sin\alpha\sin\beta$$
$$b = \cosh\gamma\sin\alpha\cos\beta - \sinh\gamma\cos\alpha\sin\beta$$
$$c = \cosh\gamma\cos\alpha\sin\beta - \sinh\gamma\sin\alpha\cos\beta$$
$$d = \cosh\gamma\sin\alpha\sin\beta + \sinh\gamma\cos\alpha\cos\beta$$

$$\begin{cases} a + d = e^{+\gamma} * \cos(\alpha - \beta) \\ b - c = e^{+\gamma} * \sin(\alpha - \beta) \\ a - d = e^{-\gamma} * \cos(\alpha + \beta) \\ b + c = e^{-\gamma} * \sin(\alpha + \beta) \end{cases}$$

$$\begin{cases} (a + d)^2 + (b - c)^2 = e^{2\gamma} \\ (a - d)^2 + (b + c)^2 = e^{-2\gamma} \end{cases}$$

$$\begin{cases} V = a^2 + b^2 + c^2 + d^2 = \cosh 2\gamma \\ U = 2(ad - bc) = \sinh 2\gamma \\ V^2 - U^2 = 1 \end{cases}$$

12.2 e−

$$\bar{z} = e^{-i\alpha - j\beta - ij\gamma} = (\underline{a} + i\underline{b}) + j(\underline{c} + i\underline{d})$$

$$\bar{z} = (\cos\alpha - i\sin\alpha)(\cos\beta - j\sin\beta)(\cosh\gamma - ij\sinh\gamma)$$
$$\bar{z} = (\cos\alpha\cos\beta - i\sin\alpha\cos\beta - j\cos\alpha\sin\beta + ij\sin\alpha\sin\beta)(\cosh\gamma - ij\sinh\gamma)$$

$$\begin{cases} \underline{a} = \cosh\gamma\cos\alpha\cos\beta - \sinh\gamma\sin\alpha\sin\beta \\ \underline{b} = -\cosh\gamma\sin\alpha\cos\beta - \sinh\gamma\cos\alpha\sin\beta \\ \underline{c} = -\cosh\gamma\cos\alpha\sin\beta - \sinh\gamma\sin\alpha\cos\beta \\ \underline{d} = \cosh\gamma\sin\alpha\sin\beta - \sinh\gamma\cos\alpha\cos\beta \end{cases}$$

$$\begin{cases} \underline{a} + \underline{d} = e^{-\gamma} * \cos(\alpha - \beta) \\ \underline{b} - \underline{c} = -e^{-\gamma} * \sin(\alpha - \beta) \\ \underline{a} - \underline{d} = e^{+\gamma} * \cos(\alpha + \beta) \\ \underline{b} + \underline{c} = -e^{+\gamma} * \sin(\alpha + \beta) \end{cases}$$

$$\begin{cases} (\underline{a} + \underline{d})^2 + (\underline{b} - \underline{c})^2 = e^{-2\gamma} \\ (\underline{a} - \underline{d})^2 + (\underline{b} + \underline{c})^2 = e^{2\gamma} \end{cases}$$

$$\begin{cases} \underline{V} = \underline{a}^2 + \underline{b}^2 + \underline{c}^2 + \underline{d}^2 = \cosh 2\gamma \\ \underline{U} = 2(\underline{a}\underline{d} - \underline{b}\underline{c}) = -\sinh 2\gamma \\ \underline{V}^2 - \underline{U}^2 = 1 \end{cases}$$

$$\underline{a} = aV - dU$$
$$\underline{b} = -bV - cU$$
$$\underline{c} = -bU - cV$$
$$\underline{d} = -aU + dV$$

13 Exponentiation

$z = (a + ib) + j(c + id)$
$\quad = R * e^{i\alpha + j\beta + ij\gamma}$
$\quad = R * e^{i(\alpha + \pi \cdot k + 2\pi \cdot m)} * e^{j(\beta + \pi \cdot k + 2\pi \cdot p)} * e^{ij\gamma}$
$\quad = R * e^{i\alpha} * e^{j\beta} * e^{ij\gamma} * e^{(\pi \cdot k + 2\pi \cdot m)} * e^{(\pi \cdot k + 2\pi \cdot p)}$

$e^{i(\alpha) + j(\beta)} = e^{i(\alpha + \pi) + j(\beta + \pi)}$
$\quad\quad\quad\quad = e^{i(\alpha + \pi) + j(\beta - \pi)}$
$\quad\quad\quad\quad = e^{i(\alpha - \pi) + j(\beta + \pi)}$
$\quad\quad\quad\quad = e^{i(\alpha - \pi) + j(\beta - \pi)}$

13.1 z^n

$z^n = R^n * e^{i(\alpha \cdot n)} * e^{j(\beta \cdot n)} * e^{ij(\gamma \cdot n)}$

13.2 $\sqrt[n]{z}$

$\sqrt[n]{z} = \sqrt[n]{R} * e^{i(\alpha/n)} * e^{j(\beta/n)} * e^{ij(\gamma/n)} * e^{i(\pi \cdot k + 2\pi \cdot m)/n} * e^{j(\pi \cdot k + 2\pi \cdot p)/n}$
$\quad k = 0, 1$
$\quad m = 0, 1, ..., n-1$
$\quad p = 0, 1, ..., n-1$

$\left(\sqrt[n]{z} \right)_{<0|0|0>} = \sqrt[n]{R} * e^{i(\alpha/n)} * e^{j(\beta/n)} * e^{ij(\gamma/n)}$
$\left(\sqrt[n]{z} \right)_{<0|m|p>} = \sqrt[n]{R} * e^{i(\alpha + 2\pi \cdot m)/n} * e^{j(\beta + 2\pi \cdot p)/n} * e^{ij(\gamma/n)}$
$\left(\sqrt[n]{z} \right)_{<1|m|p>} = \sqrt[n]{R} * e^{i(\alpha + \pi + 2\pi \cdot m)/n} * e^{j(\beta + \pi + 2\pi \cdot p)/n} * e^{ij(\gamma/n)}$

13.2.1 $\sqrt{z}$

$n = 2$

$\sqrt{z} = \left(\sqrt{R} * e^{i(\alpha/2)} * e^{j(\beta/2)} * e^{ij(\gamma/2)} \right) * e^{i(\pi \cdot k + 2\pi \cdot m)/2} * e^{j(\pi \cdot k + 2\pi \cdot p)/2}$
$\quad = R' * e^{i(\pi \cdot k + 2\pi \cdot m)/2} * e^{j(\pi \cdot k + 2\pi \cdot p)/2}$

$(\sqrt{z})_{<0|0|0>} = R'$
$(\sqrt{z})_{<0|1|0>} = R' * e^{i(\pi)} = R' * (-1)$
$(\sqrt{z})_{<0|0|1>} \doteq R' * e^{j(\pi)} = R' * (-1)$
$(\sqrt{z})_{<0|1|1>} \doteq R' * e^{i(\pi)} * e^{j(\pi)} = R'$

$(\sqrt{z})_{<1|0|0>} = R' * e^{i(\pi/2)} * e^{j(\pi/2)} = R' * (ij)$
$(\sqrt{z})_{<1|1|0>} = R' * e^{i(\pi/2 + \pi)} * e^{j(\pi/2)} = R' * (-1) * e^{i(\pi/2)} * e^{j(\pi/2)} = R' * (-ij)$
$(\sqrt{z})_{<1|0|1>} \doteq R' * e^{i(\pi/2)} * e^{j(\pi/2 + \pi)} = R' * (-1) * e^{i(\pi/2)} * e^{j(\pi/2)} = R' * (-ij)$
$(\sqrt{z})_{<1|1|1>} \doteq R' * e^{i(\pi/2 + \pi)} * e^{j(\pi/2 + \pi)} = R' * e^{i(\pi/2)} * e^{j(\pi/2)} = R' * (ij)$

$(\sqrt{z}) = R' * (\pm 1)$
$(\sqrt{z}) = R' * (\pm ij)$

$(\sqrt{z})_{<0|0|0>} = \sqrt{R} * e^{i(\alpha/2)} * e^{j(\beta/2)} * e^{ij(\gamma/2)}$
$(\sqrt{z})_{<0|1|0>} = -\sqrt{R} * e^{i(\alpha/2)} * e^{j(\beta/2)} * e^{ij(\gamma/2)}$
$(\sqrt{z})_{<1|0|0>} = \sqrt{R} * e^{i(\alpha/2 + \pi/2)} * e^{j(\beta/2 + \pi/2)} * e^{ij(\gamma/2)}$
$(\sqrt{z})_{<1|1|0>} = -\sqrt{R} * e^{i(\alpha/2 + \pi/2)} * e^{j(\beta/2 + \pi/2)} * e^{ij(\gamma/2)}$

$(\sqrt{1})_{<0|0|0>} = 1$
$(\sqrt{1})_{<0|1|0>} = -1$
$(\sqrt{1})_{<1|0|0>} = ij$
$(\sqrt{1})_{<1|1|0>} = -ij$

$(\sqrt{i})_{<0|0|0>} = e^{i(\pi/4)}$
$(\sqrt{i})_{<0|1|0>} = - e^{i(\pi/4)}$
$(\sqrt{i})_{<1|0|0>} = - e^{i(-\pi/4)} * e^{j(\pi/2)}$
$(\sqrt{i})_{<1|1|0>} = e^{i(-\pi/4)} * e^{j(\pi/2)}$

$(\sqrt{j})_{<0|0|0>} = e^{j(\pi/4)}$
$(\sqrt{j})_{<0|1|0>} = - e^{j(\pi/4)}$
$(\sqrt{j})_{<1|0|0>} = - e^{i(\pi/2)} * e^{j(-\pi/4)}$
$(\sqrt{j})_{<1|1|0>} = e^{i(\pi/2)} * e^{j(-\pi/4)}$

$$\left(\sqrt{ij}\right)_{<0|0|0>} = \quad e^{i(\pi/4)} \quad * \, e^{j(\pi/4)}$$
$$\left(\sqrt{ij}\right)_{<0|1|0>} = - \, e^{i(\pi/4)} \quad * \, e^{j(\pi/4)}$$
$$\left(\sqrt{ij}\right)_{<1|0|0>} = \quad e^{i(-\pi/4)} * \, e^{j(-\pi/4)}$$
$$\left(\sqrt{ij}\right)_{<1|1|0>} = - \, e^{i(-\pi/4)} * \, e^{j(-\pi/4)}$$

13.2.2 $\sqrt[3]{z}$

$n = 3$

$$\sqrt[3]{z} = \left(\sqrt[3]{R} * e^{i(\alpha/3)} * e^{j(\beta/3)} * e^{ij(\gamma/3)}\right) * e^{i(\pi \cdot k + 2\pi \cdot m)/3} * e^{j(\pi \cdot k + 2\pi \cdot p)/3}$$
$$= R' * e^{i(\pi \cdot k + 2\pi \cdot m)/3} * e^{j(\pi \cdot k + 2\pi \cdot p)/3}$$

$$e^{i(\pi/3)}$$
$$e^{i(2\pi/3)} = e^{i(3\pi/3 - \pi/3)} \quad = (-1) * e^{i(-\pi/3)}$$
$$e^{i(3\pi/3)} \qquad\qquad\quad = (-1)$$
$$e^{i(4\pi/3)} = e^{i(3\pi/3 + \pi/3)} \quad = (-1) * e^{i(\pi/3)}$$
$$e^{i(5\pi/3)} = e^{i(3\pi/3 + 2\pi/3)} = (-1) * e^{i(2\pi/3)} = e^{i(-\pi/3)}$$

$$\left(\sqrt[3]{z}\right)_{<0|0|0>} = R'$$
$$\left(\sqrt[3]{z}\right)_{<0|1|0>} = R' * e^{i(2\pi/3)} = R' * (-1) * e^{i(-\pi/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|2|0>} = R' * e^{i(4\pi/3)} = R' * (-1) * e^{i(\pi/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|0|1>} = R' * e^{j(2\pi/3)} = R' * (-1) * e^{j(-\pi/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|1|1>} = R' * e^{i(2\pi/3)} * e^{j(2\pi/3)} = R' * e^{i(-\pi/3)} * e^{j(-\pi/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|2|1>} = R' * e^{i(4\pi/3)} * e^{j(2\pi/3)} = R' * (-1) * e^{i(\pi/3)} * e^{j(2\pi/3)} = R' * e^{i(\pi/3)} * e^{j(-\pi/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|0|2>} = R' * e^{j(4\pi/3)} = R' * (-1) * e^{j(\pi/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|1|2>} = R' * e^{i(2\pi/3)} * e^{j(4\pi/3)} = R' * (-1) * e^{i(2\pi/3)} * e^{j(\pi/3)} = R' * e^{i(-\pi/3)} * e^{j(\pi/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|2|2>} = R' * e^{i(4\pi/3)} * e^{j(4\pi/3)} = R' * e^{i(\pi/3)} * e^{j(\pi/3)}$$

$$\left(\sqrt[3]{z}\right)_{<1|0|0>} \doteq R' * e^{i(\pi/3)} * e^{j(\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|2|2>}$$
$$\left(\sqrt[3]{z}\right)_{<1|1|0>} \doteq R' * e^{i(\pi/3 + 2\pi/3)} * e^{j(\pi/3)} = R' * e^{i(3\pi/3)} * e^{j(\pi/3)} = R' * (-1) * e^{j(\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|0|2>}$$
$$\left(\sqrt[3]{z}\right)_{<1|2|0>} \doteq R' * e^{i(\pi/3 + 4\pi/3)} * e^{j(\pi/3)} = R' * e^{i(5\pi/3)} * e^{j(\pi/3)} = R' * (-1) * e^{i(2\pi/3)} * e^{j(\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|1|2>}$$
$$\left(\sqrt[3]{z}\right)_{<1|0|1>} \doteq R' * e^{i(\pi/3)} * e^{j(\pi/3 + 2\pi/3)} = R' * e^{i(\pi/3)} * e^{j(3\pi/3)} = R' * (-1) * e^{i(\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|2|0>}$$
$$\left(\sqrt[3]{z}\right)_{<1|1|1>} \doteq R' * e^{i(\pi/3 + 2\pi/3)} * e^{j(\pi/3 + 2\pi/3)} = R' * e^{i(3\pi/3)} * e^{j(3\pi/3)} = R' = \left(\sqrt[3]{z}\right)_{<0|0|0>}$$
$$\left(\sqrt[3]{z}\right)_{<1|2|1>} \doteq R' * e^{i(\pi/3 + 4\pi/3)} * e^{j(\pi/3 + 2\pi/3)} = R' * e^{i(5\pi/3)} * e^{j(3\pi/3)} = R' * e^{i(2\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|1|0>}$$
$$\left(\sqrt[3]{z}\right)_{<1|0|2>} \doteq R' * e^{i(\pi/3)} * e^{j(\pi/3 + 4\pi/3)} = R' * e^{i(\pi/3)} * e^{j(5\pi/3)} = R' * (-1) * e^{i(\pi/3)} * e^{j(2\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|2|1>}$$
$$\left(\sqrt[3]{z}\right)_{<1|1|2>} \doteq R' * e^{i(\pi/3 + 2\pi/3)} * e^{j(\pi/3 + 4\pi/3)} = R' * e^{i(3\pi/3)} * e^{j(5\pi/3)} = R' * e^{j(2\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|0|1>}$$
$$\left(\sqrt[3]{z}\right)_{<1|2|2>} \doteq R' * e^{i(\pi/3 + 4\pi/3)} * e^{j(\pi/3 + 4\pi/3)} = R' * e^{i(5\pi/3)} * e^{j(5\pi/3)} = R' * e^{i(2\pi/3)} * e^{j(2\pi/3)} = \left(\sqrt[3]{z}\right)_{<0|1|1>}$$

$$\left(\sqrt[3]{z}\right)_{<0|0|0>} = \quad \sqrt[3]{R} * e^{i(\alpha/3)} * e^{j(\beta/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|1|0>} = - \, \sqrt[3]{R} * e^{i(\alpha/3 - \pi/3)} * e^{j(\beta/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|2|0>} = - \, \sqrt[3]{R} * e^{i(\alpha/3 + \pi/3)} * e^{j(\beta/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|0|1>} = - \, \sqrt[3]{R} * e^{i(\alpha/3)} * e^{j(\beta/3 - \pi/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|0|2>} = - \, \sqrt[3]{R} * e^{i(\alpha/3)} * e^{j(\beta/3 + \pi/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|1|1>} = \quad \sqrt[3]{R} * e^{i(\alpha/3 - \pi/3)} * e^{j(\beta/3 - \pi/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|1|2>} = \quad \sqrt[3]{R} * e^{i(\alpha/3 - \pi/3)} * e^{j(\beta/3 + \pi/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|2|1>} = \quad \sqrt[3]{R} * e^{i(\alpha/3 + \pi/3)} * e^{j(\beta/3 - \pi/3)} * e^{ij(\gamma/3)}$$
$$\left(\sqrt[3]{z}\right)_{<0|2|2>} = \quad \sqrt[3]{R} * e^{i(\alpha/3 + \pi/3)} * e^{j(\beta/3 + \pi/3)} * e^{ij(\gamma/3)}$$

14 Elementary functions

$$\sin(\ i \cdot\phi) = i \cdot \sinh(\phi) \qquad \sinh(\ i \cdot\phi) = i \cdot \sin(\phi)$$
$$\sin(\ j \cdot\phi) = j \cdot \sinh(\phi) \qquad \sinh(\ j \cdot\phi) = j \cdot \sin(\phi)$$
$$\cos(\ i \cdot\phi) = \cosh(\phi) \qquad \cosh(\ i \cdot\phi) = \cos(\phi)$$
$$\cos(\ j \cdot\phi) = \cosh(\phi) \qquad \cosh(\ j \cdot\phi) = \cos(\phi)$$
$$\sin(\ ij \cdot\phi) = ij \cdot \sin(\phi) \qquad \sinh(\ ij \cdot\phi) = ij \cdot \sinh(\phi)$$
$$\cos(\ ij \cdot\phi) = \cos(\phi) \qquad \cosh(\ ij \cdot\phi) = \cosh(\phi)$$

14.1 sin

$$\sin(a+ib) = \underline{a} + i\underline{b}$$
$$\underline{a} = \sin(a) * \cosh(b)$$
$$\underline{b} = \cos(a) * \sinh(b)$$

$$\sin(z) = \sin(a+ib+jc+ijd) = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d}$$
$$\underline{a} = \quad 1/2 * \sin\ (a-d) * \cosh\ (b+c) + 1/2 * \sin\ (a+d) * \cosh\ (b-c)$$
$$\underline{b} = \quad 1/2 * \cos\ (a-d) * \sinh\ (b+c) + 1/2 * \cos\ (a+d) * \sinh\ (b-c)$$
$$\underline{c} = \quad 1/2 * \cos\ (a-d) * \sinh\ (b+c) - 1/2 * \cos\ (a+d) * \sinh\ (b-c)$$
$$\underline{d} = -1/2 * \sin\ (a-d) * \cosh\ (b+c) + 1/2 * \sin\ (a+d) * \cosh\ (b-c)$$

14.2 cos

$$\cos(a+ib) = \underline{a} + i\underline{b}$$
$$\underline{a} = \quad \cos(a) * \cosh(b)$$
$$\underline{b} = -\sin(a) * \sinh(b)$$

$$\cos(z) = \cos(a+ib+jc+ijd) = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d}$$
$$\underline{a} = \quad 1/2 * \cos\ (a-d) * \cosh\ (b+c) + 1/2 * \cos\ (a+d) * \cosh\ (b-c)$$
$$\underline{b} = -1/2 * \sin\ (a-d) * \sinh\ (b+c) - 1/2 * \sin\ (a+d) * \sinh\ (b-c)$$
$$\underline{c} = -1/2 * \sin\ (a-d) * \sinh\ (b+c) + 1/2 * \sin\ (a+d) * \sinh\ (b-c)$$
$$\underline{d} = -1/2 * \cos\ (a-d) * \cosh\ (b+c) + 1/2 * \cos\ (a+d) * \cosh\ (b-c)$$

14.3 sinh

$$\sinh(a+ib) = \underline{a} + i\underline{b}$$
$$\underline{a} = \sinh(a) * \cos(b)$$
$$\underline{b} = \cosh(a) * \sin(b)$$

$$\sinh(z) = \sinh(a+ib+jc+ijd) = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d}$$
$$\underline{a} = \quad 1/2 * \sinh\ (a-d) * \cos\ (b+c) + 1/2 * \sinh\ (a+d) * \cos\ (b-c)$$
$$\underline{b} = \quad 1/2 * \cosh\ (a-d) * \sin\ (b+c) + 1/2 * \cosh\ (a+d) * \sin\ (b-c)$$
$$\underline{c} = \quad 1/2 * \cosh\ (a-d) * \sin\ (b+c) - 1/2 * \cosh\ (a+d) * \sin\ (b-c)$$
$$\underline{d} = -1/2 * \sinh\ (a-d) * \cos\ (b+c) + 1/2 * \sinh\ (a+d) * \cos\ (b-c)$$

14.4 cosh

$$\cosh(a+ib) = \underline{a} + i\underline{b}$$
$$\underline{a} = \cosh(a) * \cos(b)$$
$$\underline{b} = \sinh(a) * \sin(b)$$

$$\cosh(z) = \cosh(a+ib+jc+ijd) = \underline{a} + i\underline{b} + j\underline{c} + ij\underline{d}$$
$$\underline{a} = \quad 1/2 * \cosh\ (a-d) * \cos\ (b+c) + 1/2 * \cosh\ (a+d) * \cos\ (b-c)$$
$$\underline{b} = \quad 1/2 * \sinh\ (a-d) * \sin\ (b+c) + 1/2 * \sinh\ (a+d) * \sin\ (b-c)$$
$$\underline{c} = \quad 1/2 * \sinh\ (a-d) * \sin\ (b+c) - 1/2 * \sinh\ (a+d) * \sin\ (b-c)$$
$$\underline{d} = -1/2 * \cosh\ (a-d) * \cos\ (b+c) + 1/2 * \cosh\ (a+d) * \cos\ (b-c)$$

14.5 tan & Co.

$$\tan(z) \quad = \sin(z) \,/\, \cos(z)$$
$$\cot(z) \quad = \cos(z) \,/\, \sin(z)$$

$$\tanh(z) \quad = \sinh(z) \,/\, \cosh(z)$$
$$\coth(z) \quad = \cosh(z) \,/\, \sinh(z)$$

$$\sec(z) \quad = 1 \,/\, \cos(z)$$
$$\operatorname{cosec}(z) \quad = 1 \,/\, \sin(z)$$

$$\operatorname{sech}(z) \quad = 1 \,/\, \cosh(z)$$
$$\operatorname{cosech}(z) = 1 \,/\, \sinh(z)$$

14.6 inverse functions $< i >$

$$\operatorname{asin}(z) \quad = \pi/2 - i \cdot \ln\left(z + \sqrt{z^2 - 1}\right) = -i \cdot \ln\left(iz + \sqrt{1 - z^2}\right)$$

$$\operatorname{acos}(z) \quad = i \cdot \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\operatorname{asinh}(z) = \ln\left(z + \sqrt{z^2 + 1}\right)$$

$$\operatorname{acosh}(z) = \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\operatorname{asec}(z) \quad = \operatorname{acos}\left(\frac{1}{z}\right) \quad = i \cdot \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right)$$

$$\operatorname{acosec}(z) \quad = \operatorname{asin}\left(\frac{1}{z}\right) \quad = \frac{\pi}{2} - i \cdot \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right)$$

$$\operatorname{asech}(z) \quad = \operatorname{acosh}\left(\frac{1}{z}\right) = \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right)$$

$$\operatorname{acosech}(z) = \operatorname{asinh}\left(\frac{1}{z}\right) = \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} + 1}\right)$$

$$\operatorname{atan}(z) \quad = -\frac{i}{2} \ln\left(\frac{i - z}{i + z}\right) = -\frac{i}{2} \ln\left(\frac{1 + iz}{1 - iz}\right)$$

$$\operatorname{acot}(z) \quad = -\frac{i}{2} \ln\left(\frac{z + i}{z - i}\right) = -\frac{i}{2} \ln\left(\frac{iz - 1}{iz + 1}\right)$$

$$\operatorname{atanh}(z) = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z}\right)$$

$$\operatorname{acoth}(z) = \frac{1}{2} \ln\left(\frac{z + 1}{z - 1}\right)$$

14.7 normalized functions

$$nF(z) \equiv F(z) \,/\, |F(z)|$$

$$\operatorname{nsin}(z) = \sin(z) \,/\, |\sin(z)|$$
$$\operatorname{ncos}(z) = \cos(z) \,/\, |\cos(z)|$$
$$\operatorname{ntan}(z) = \tan(z) \,/\, |\tan(z)|$$
$$\operatorname{ncot}(z) = \cot(z) \,/\, |\cot(z)|$$

...

14.8   exp

$$\exp(z) \quad = e^z = e^{a+ib+jc+ijd} = e^a * e^{ib+jc+ijd}$$
$$\mathrm{nexp}(z) = e^z \ / \ |e^z| = e^{ib+jc+ijd}$$

14.9   ln

$$\ln(z) \quad = \ln(R * e^{i\alpha+j\beta+ij\gamma}) = \ln(R) + i\alpha + j\beta + ij\gamma$$
$$\mathrm{Ln}(z) = \ln(R) + i(\alpha + \pi \cdot k + 2\pi \cdot m) + j(\beta + \pi \cdot k + 2\pi \cdot p) + ij\gamma$$
$$k = 0, 1$$
$$m = 0, 1, ...$$
$$p = 0, 1, ...$$

$$\mathrm{Ln}(z)_{<0|0|0>} \ = \ln(R) + i\alpha + j\beta + ij\gamma$$
$$\mathrm{Ln}(z)_{<0|m|p>} = \ln(R) + i(\alpha + 2\pi \cdot m) + j(\beta + 2\pi \cdot p) + ij\gamma$$
$$\mathrm{Ln}(z)_{<1|m|p>} = \ln(R) + i(\alpha + \pi + 2\pi \cdot m) + j(\beta + \pi + 2\pi \cdot p) + ij\gamma$$

15 Alternative notation

$z = a + ib + jc + ijd = R * e^{i\alpha + j\beta + ij\gamma}$

$z = a_0 \cdot (1 + ij) / 2 + b_0 \cdot (i - j) / 2 + c_0 \cdot (1 - ij) / 2 + d_0 \cdot (i + j) / 2$

$(1 + ij) * (1 + ij) = 1 + ij + ij + 1 \quad = \quad 2 \cdot (1 + ij)$
$(1 + ij) * (i - j) = i - j - j + i \quad = \quad 2 \cdot (i - j)$
$(1 + ij) * (1 - ij) = 1 - ij + ij - 1 \quad = \quad 0$
$(1 + ij) * (i + j) = i + j - j - i \quad = \quad 0$

$(i - j) * (1 + ij) = i - j - j + i \quad = \quad 2 \cdot (i - j)$
$(i - j) * (i - j) = -1 - ij - ij - 1 = -2 \cdot (1 + ij)$
$(i - j) * (1 - ij) = i + j - j - i \quad = \quad 0$
$(i - j) * (i + j) = -1 + ij - ij + 1 = \quad 0$

$(1 - ij) * (1 + ij) = 1 + ij - ij - 1 \quad = \quad 0$
$(1 - ij) * (i - j) = i - j + j - i \quad = \quad 0$
$(1 - ij) * (1 - ij) = 1 - ij - ij + 1 \quad = \quad 2 \cdot (1 - ij)$
$(1 - ij) * (i + j) = i + j + j + i \quad = \quad 2 \cdot (i + j)$

$(i + j) * (1 + ij) = i - j + j - i \quad = \quad 0$
$(i + j) * (i - j) = -1 - ij_i j + 1 \quad = \quad 0$
$(i + j) * (1 - ij) = i + j + j + i \quad = \quad 2 \cdot (i + j)$
$(i + j) * (i + j) = -1 + ij + ij - 1 = -2 \cdot (1 - ij)$

15.1

$$\begin{cases} a = (a_0 + c_0) / 2 \\ b = (d_0 + b_0) / 2 \\ c = (d_0 - b_0) / 2 \\ d = (a_0 - c_0) / 2 \end{cases}$$

$$\begin{cases} a_0 = a + d \\ b_0 = b - c \\ c_0 = a - d \\ d_0 = b + c \end{cases}$$

15.2

$$R = \sqrt[4]{(a_0^2 + b_0^2) * (c_0^2 + d_0^2)} = \sqrt[4]{\left((a + d)^2 + (b - c)^2\right) * \left((a - d)^2 + (b + c)^2\right)}$$

$$\gamma = \frac{1}{4} \ln \frac{(a_0^2 + b_0^2)}{(c_0^2 + d_0^2)} = \frac{1}{4} \ln \frac{(a + d)^2 + (b - c)^2}{(a - d)^2 + (b + c)^2}$$

$$\alpha = \frac{1}{2} \left[\arg(a_0, b_0) + \arg(c_0, d_0)\right] = \frac{1}{2} \left[\arg(a + d, b - c) + \arg(a - d, b + c)\right]$$

$$\beta = \frac{1}{2} \left[\arg(c_0, d_0) - \arg(a_0, b_0)\right] = \frac{1}{2} \left[\arg(a - d, b + c) - \arg(a + d, b - c)\right]$$

$$\arg(x, y) = \arctan(y, x) = \arctan(y : x) = \arctan(y/x)$$

15.3

$$\begin{cases} a_0 = a + d = R * e^{+\gamma} * \cos(\alpha - \beta) \\ b_0 = b - c = R * e^{+\gamma} * \sin(\alpha - \beta) \\ c_0 = a - d = R * e^{-\gamma} * \cos(\alpha + \beta) \\ d_0 = b + c = R * e^{-\gamma} * \sin(\alpha + \beta) \end{cases}$$

$$\begin{cases} R * e^{+\gamma} = \sqrt{a_0^2 + b_0^2} = \sqrt{(a + d)^2 + (b - c)^2} \\ R * e^{-\gamma} = \sqrt{c_0^2 + d_0^2} = \sqrt{(a - d)^2 + (b + c)^2} \end{cases}$$

$$\begin{cases} \alpha - \beta = \arg(a_0, b_0) = \arg(a + d, b - c) \\ \alpha + \beta = \arg(c_0, d_0) = \arg(a - d, b + c) \end{cases}$$

16 Dimension reduction

$w = (a + ib) + j \cdot (c + id) = r * e^{i\phi} + j \cdot \rho * e^{i\psi}$
$\phi = \psi \Rightarrow$

$w = a + i \cdot b + j \cdot c + ij \cdot d = (x, y, z)$
$R^2 = a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2$

16.1 xy:z$<\cdot\cdot>$

$z' = z \,/\, \sqrt{x^2 + y^2}$

16.1.1 xy:z$<$ab$>$

$w = (x + iy) + j \cdot (x + iy) \cdot z' = (x + iy) + j \cdot (x + iy) \cdot z \,/\, \sqrt{x^2 + y^2}$
$w = x + i \cdot y + j \cdot x \cdot z \,/\, \sqrt{x^2 + y^2} + ij \cdot y \cdot z \,/\, \sqrt{x^2 + y^2} = a + i \cdot b + j \cdot c + ij \cdot d$

$a = x$	$x = a$
$b = y$	$y = b$
$c = x \cdot z \,/\, \sqrt{x^2 + y^2}$	$z = c \cdot \sqrt{a^2 + b^2} \,/\, a$
$d = y \cdot z \,/\, \sqrt{x^2 + y^2}$	$z = d \cdot \sqrt{a^2 + b^2} \,/\, b$

$x = y \Rightarrow \quad a = x \quad b = y \quad c = \text{sign}(x) \cdot z/\sqrt{2} \quad d = \text{sign}(y) \cdot z/\sqrt{2}$

$\tan\phi = \tan\psi = y \,/\, x$

16.1.2 xy:z$<$cd$>$

$w = (x + iy) \cdot z' + j \cdot (x + iy) = (x + iy) \cdot z \,/\, \sqrt{x^2 + y^2} + j \cdot (x + iy)$
$w = x \cdot z \,/\, \sqrt{x^2 + y^2} + i \cdot y \cdot z \,/\, \sqrt{x^2 + y^2} + jx + ijy = a + i \cdot b + j \cdot c + ij \cdot d$

$a = x \cdot z \,/\, \sqrt{x^2 + y^2}$	$x = c$
$b = y \cdot z \,/\, \sqrt{x^2 + y^2}$	$y = d$
$c = x$	$z = a \cdot \sqrt{c^2 + d^2} \,/\, c$
$d = y$	$z = b \cdot \sqrt{c^2 + d^2} \,/\, d$

$x = y \Rightarrow \quad a = \text{sign}(x) \cdot z/\sqrt{2} \quad b = \text{sign}(y) \cdot z/\sqrt{2} \quad c = x \quad d = y$

$\tan\phi = \tan\psi = y \,/\, x$

16.1.3 xy:z$<$ba$>$

$w = (y + ix) + j \cdot (y + ix) \cdot z' = (y + ix) + j \cdot (y + ix) \cdot z \,/\, \sqrt{x^2 + y^2}$
$w = y + i \cdot x + j \cdot y \cdot z \,/\, \sqrt{x^2 + y^2} + ij \cdot x \cdot z \,/\, \sqrt{x^2 + y^2} = a + i \cdot b + j \cdot c + ij \cdot d$

$a = y$	$x = b$
$b = x$	$y = a$
$c = y \cdot z \,/\, \sqrt{x^2 + y^2}$	$z = c \cdot \sqrt{a^2 + b^2} \,/\, a$
$d = x \cdot z \,/\, \sqrt{x^2 + y^2}$	$z = d \cdot \sqrt{a^2 + b^2} \,/\, b$

$x = y \Rightarrow \quad a = y \quad b = x \quad c = \text{sign}(y) \cdot z/\sqrt{2} \quad d = \text{sign}(x) \cdot z/\sqrt{2}$

$\tan\phi = \tan\psi = x \,/\, y$

16.1.4 xy:z$<$dc$>$

$w = (y + ix) \cdot z' + j \cdot (y + ix) = (y + ix) \cdot z \,/\, \sqrt{x^2 + y^2} + j \cdot (y + ix)$
$w = y \cdot z \,/\, \sqrt{x^2 + y^2} + i \cdot x \cdot z \,/\, \sqrt{x^2 + y^2} + jy + ijx = a + i \cdot b + j \cdot c + ij \cdot d$

$a = y \cdot z \,/\, \sqrt{x^2 + y^2}$	$x = d$
$b = x \cdot z \,/\, \sqrt{x^2 + y^2}$	$y = c$
$c = y$	$z = a \cdot \sqrt{c^2 + d^2} \,/\, c$
$d = x$	$z = b \cdot \sqrt{c^2 + d^2} \,/\, d$

$x = y \Rightarrow \quad a = \text{sign}(y) \cdot z/\sqrt{2} \quad b = \text{sign}(x) \cdot z/\sqrt{2} \quad c = y \quad d = x$

$\tan\phi = \tan\psi = x \,/\, y$

16.1.5 xy:z<ac>

$$w = (x + jy) + i \cdot (x + jy) \cdot z' = (x + jy) + i \cdot (x + jy) \cdot z \,/\, \sqrt{x^2 + y^2}$$
$$w = x + i \cdot x \cdot z \,/\, \sqrt{x^2 + y^2} + j \cdot y + ij \cdot y \cdot z \,/\, \sqrt{x^2 + y^2} = a + i \cdot b + j \cdot c + ij \cdot d$$

$$a = x \qquad\qquad\qquad x = a$$
$$b = x \cdot z \,/\, \sqrt{x^2 + y^2} \qquad y = c$$
$$c = y \qquad\qquad\qquad z = b \cdot \sqrt{a^2 + b^2} \,/\, a$$
$$d = y \cdot z \,/\, \sqrt{x^2 + y^2} \qquad z = d \cdot \sqrt{a^2 + b^2} \,/\, c$$

$$x = y \Rightarrow \quad a = x \quad b = \text{sign}(x) \cdot z/\sqrt{2} \quad c = y \quad d = \text{sign}(y) \cdot z/\sqrt{2}$$

$$\tan\phi = \tan\psi = z \,/\, \sqrt{x^2 + y^2}$$

16.1.6 xy:z<ca>

$$w = (y + jx) + i \cdot (y + jx) \cdot z' = (y + jx) + i \cdot (y + jx) \cdot z \,/\, \sqrt{x^2 + y^2}$$
$$w = y + i \cdot y \cdot z \,/\, \sqrt{x^2 + y^2} + j \cdot x + ij \cdot x \cdot z \,/\, \sqrt{x^2 + y^2} = a + i \cdot b + j \cdot c + ij \cdot d$$

$$a = y \qquad\qquad\qquad x = c$$
$$b = y \cdot z \,/\, \sqrt{x^2 + y^2} \qquad y = a$$
$$c = x \qquad\qquad\qquad z = b \cdot \sqrt{a^2 + b^2} \,/\, a$$
$$d = x \cdot z \,/\, \sqrt{x^2 + y^2} \qquad z = d \cdot \sqrt{a^2 + b^2} \,/\, c$$

$$x = y \Rightarrow \quad a = y \quad b = \text{sign}(y) \cdot z/\sqrt{2} \quad c = x \quad d = \text{sign}(x) \cdot z/\sqrt{2}$$

$$\tan\phi = \tan\psi = z \,/\, \sqrt{x^2 + y^2}$$

16.1.7 xy:z<bd>

$$w = (x + jy) \cdot z' + i \cdot (x + jy) = (x + jy) \cdot z \,/\, \sqrt{x^2 + y^2} + i \cdot (x + jy)$$
$$w = x \cdot z \,/\, \sqrt{x^2 + y^2} + ix + j \cdot y \cdot z \,/\, \sqrt{x^2 + y^2} + ijy = a + i \cdot b + j \cdot c + ij \cdot d$$

$$a = x \cdot z \,/\, \sqrt{x^2 + y^2} \qquad x = b$$
$$b = x \qquad\qquad\qquad y = d$$
$$c = y \cdot z \,/\, \sqrt{x^2 + y^2} \qquad z = a \cdot \sqrt{c^2 + d^2} \,/\, b$$
$$d = y \qquad\qquad\qquad z = c \cdot \sqrt{c^2 + d^2} \,/\, d$$

$$x = y \Rightarrow \quad a = \text{sign}(x) \cdot z/\sqrt{2} \quad b = x \quad c = \text{sign}(y) \cdot z/\sqrt{2} \quad d = y$$

$$\tan\phi = \tan\psi = \sqrt{x^2 + y^2} \,/\, z$$

16.1.8 xy:z<db>

$$w = (y + jx) \cdot z' + i \cdot (y + jx) = (y + jx) \cdot z \,/\, \sqrt{x^2 + y^2} + i \cdot (y + jx)$$
$$w = y \cdot z \,/\, \sqrt{x^2 + y^2} + iy + j \cdot x \cdot z \,/\, \sqrt{x^2 + y^2} + ijx = a + i \cdot b + j \cdot c + ij \cdot d$$

$$a = y \cdot z \,/\, \sqrt{x^2 + y^2} \qquad x = d$$
$$b = y \qquad\qquad\qquad y = b$$
$$c = x \cdot z \,/\, \sqrt{x^2 + y^2} \qquad z = a \cdot \sqrt{c^2 + d^2} \,/\, b$$
$$d = x \qquad\qquad\qquad z = c \cdot \sqrt{c^2 + d^2} \,/\, d$$

$$x = y \Rightarrow \quad a = \text{sign}(y) \cdot z/\sqrt{2} \quad b = y \quad c = \text{sign}(x) \cdot z/\sqrt{2} \quad d = x$$

$$\tan\phi = \tan\psi = \sqrt{x^2 + y^2} \,/\, z$$

16.2 yz:x$<\cdot\cdot>$

$x' = x \,/\, \sqrt{y^2 + z^2}$

$$yz : x < ab > \quad w = (y + iz) + j \cdot (y + iz) \cdot x'$$
$$yz : x < cd > \quad w = (y + iz) \cdot x' + j \cdot (y + iz)$$
$$yz : x < ba > \quad w = (z + iy) + j \cdot (z + iy) \cdot x'$$
$$yz : x < dc > \quad w = (z + iy) \cdot x' + j \cdot (z + iy)$$
$$yz : x < ac > \quad w = (y + jz) + i \cdot (y + jz) \cdot x'$$
$$yz : x < ca > \quad w = (z + jy) + i \cdot (z + jy) \cdot x'$$
$$yz : x < bd > \quad w = (y + jz) \cdot x' + i \cdot (y + jz)$$
$$yz : x < db > \quad w = (z + jy) \cdot x' + i \cdot (z + jy)$$

16.3 zx:y$<\cdot\cdot>$

$y' = y \,/\, \sqrt{x^2 + z^2}$

$$zx : y < ab > \quad w = (x + iz) + j \cdot (x + iz) \cdot y'$$
$$zx : y < cd > \quad w = (x + iz) \cdot y' + j \cdot (x + iz)$$
$$zx : y < ba > \quad w = (z + ix) + j \cdot (z + ix) \cdot y'$$
$$zx : y < dc > \quad w = (z + ix) \cdot y' + j \cdot (z + ix)$$
$$zx : y < ac > \quad w = (x + jz) + i \cdot (x + jz) \cdot y'$$
$$zx : y < ca > \quad w = (z + jx) + i \cdot (z + jx) \cdot y'$$
$$zx : y < bd > \quad w = (x + jz) \cdot y' + i \cdot (x + jz)$$
$$zx : y < db > \quad w = (z + jx) \cdot y' + i \cdot (z + jx)$$

17 Dimension reduction: xy:z<ab>

$w = (a + i \cdot b) + j \cdot (c + i \cdot d) = r * e^{i\phi} + j \cdot \rho * e^{i\psi}$
$\phi = \psi$

$w = x + i \cdot y + j \cdot x \cdot z \, / \, \sqrt{x^2 + y^2} + ij \cdot y \cdot z \, / \, \sqrt{x^2 + y^2}$

$$
\begin{array}{ll}
a = x & x = a \\
b = y & y = b \\
c = x \cdot z \, / \, \sqrt{x^2 + y^2} & z = c \cdot \sqrt{a^2 + b^2} \, / \, a \\
d = y \cdot z \, / \, \sqrt{x^2 + y^2} & z = d \cdot \sqrt{a^2 + b^2} \, / \, b
\end{array}
$$

$x = y \neq 0 \Rightarrow \quad a = x \quad b = y \quad c = \text{sign}(x) \cdot z / \sqrt{2} \quad d = \text{sign}(y) \cdot z / \sqrt{2}$

$x = y = 0 \Rightarrow \quad a = 0 \quad b = 0 \quad c \stackrel{\text{def}}{=} z / \sqrt{2} \qquad\qquad d \stackrel{\text{def}}{=} z / \sqrt{2}$

$d = c \cdot \dfrac{y}{x}$

$a \cdot d = b \cdot c = \dfrac{x \cdot y \cdot z}{\sqrt{x^2 + y^2}}$

$a + d = (x \cdot \sqrt{x^2 + y^2} + y \cdot z) \, / \, \sqrt{x^2 + y^2}$
$a - d = (x \cdot \sqrt{x^2 + y^2} - y \cdot z) \, / \, \sqrt{x^2 + y^2}$
$b + c = (y \cdot \sqrt{x^2 + y^2} + x \cdot z) \, / \, \sqrt{x^2 + y^2}$
$b - c = (y \cdot \sqrt{x^2 + y^2} - x \cdot z) \, / \, \sqrt{x^2 + y^2}$

$\tan \phi = \tan \psi = y \, / \, x$

17.1 Conjugate

$w \quad = (a + ib) + j(c + id) = a + ib + jc + ijd$
$\bar{w}_{ij} = (a - ib) - j(c - id) = a - ib - jc + ijd$

$w * \bar{w}_{ij} = R^2 = a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2$

17.2 Exponential form

$w = (a + ib) + j(c + id) = r * e^{i\phi} + j \cdot \rho * e^{i\psi}$
$w = |w| * e^{i\alpha + j\beta + ij\gamma} = R * e^{i\alpha + j\beta + ij\gamma}$

$\phi = \psi$
$\gamma = 0$

$w = |w| * e^{i\alpha + j\beta} = R * e^{i\alpha + j\beta} = R * (\cos\alpha + i\sin\alpha) * (\cos\beta + j\sin\beta)$

$a = R \cdot \cos\alpha \cdot \cos\beta$
$b = R \cdot \sin\alpha \cdot \cos\beta$
$c = R \cdot \cos\alpha \cdot \sin\beta$
$d = R \cdot \sin\alpha \cdot \sin\beta$

$R^2 = a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2$

$$
\begin{cases}
\tan(\alpha - \beta) = \dfrac{b - c}{a + d} = \dfrac{y \cdot \sqrt{x^2 + y^2} - x \cdot z}{x \cdot \sqrt{x^2 + y^2} + y \cdot z} = \arg(a + d, b - c) \\[3mm]
\tan(\alpha + \beta) = \dfrac{b + c}{a - d} = \dfrac{y \cdot \sqrt{x^2 + y^2} + x \cdot z}{x \cdot \sqrt{x^2 + y^2} - y \cdot z} = \arg(a - d, b + c)
\end{cases}
$$

$\alpha = \dfrac{1}{2} \left[\arg(a + d, b - c) + \arg(a - d, b + c) \right]$

$\beta = \dfrac{1}{2} \left[\arg(a - d, b + c) - \arg(a + d, b - c) \right]$

$\arg(x, y) = \arctan(y, x) = \arctan(y : x) = \arctan(y/x)$